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**LETTER TO THE EDITOR**

# Discrete version of the Chazy class III equation

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**Abstract.** We study the discretization of the Chazy class III equation by two means: a discrete Painlevé test, and the preservation of a two-parameter solution to the continuous equation. In this way we achieve an optimal discretization scheme.

## 1. Introduction

The Chazy class III equation [1]

$$C \equiv u''' - 2\,u\,u'' + 3\,u'^2 = 0 \tag{1}$$

is such that the only singularity of its general solution is a movable noncritical natural boundary, a circle whose centre and radius depend on the three initial conditions of the Cauchy problem. Thus, equation (1) has the Painlevé property.

Our aim is to obtain a 'most faithful' finite-difference representation of this equation. To do so, we shall first explain our method of getting the discretizing ansätze for differential equations. Second, we will demand that the discrete equation possess the 'discrete Painlevé property' defined in the sense of Conte and Musette [2]. Finally, we will examine the preservation of a two-parameter solution to equation (1).

## 2. Faithful discretization of an ODE

Given a (continuous) $N$th-order differential equation

$$\forall\ x \qquad \mathcal{E}(x, u(x), \ldots, u^{(N)}(x)) = 0 \tag{2}$$

of degree $m$ in $u^{(N)}$, a discretization scheme may be called a *(faithful) discrete version* of (2) if it satisfies the four conditions:

(i) It is an $N$th-order finite-difference equation, i.e. an iteration relation between $N+1$ values of the unknown function $u$ taken at points in an arithmetic sequence:

$$\forall\ x,\ \forall\ h \qquad E(x, h, u(x + k_0\,h), \ldots, u(x + (k_0 + N)\,h)) = 0 \tag{3}$$

The parameter $h$ is called the *step* of the finite-difference equation; the constant $k_0$ is an origin whose utility will become clear later.

(ii) *(If equation (2) has the Painlevé property.)* It is of degree $m$ in the first and last term, $u(x + k_0\,h)$ and $u(x + (k_0 + N)\,h)$, whose presence is dictated by the highest-order derivative.

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(iii) It is invariant under $(k_0, h) \mapsto (-k_0, -h)$. The reason for this condition is that the step $h$ in equation (3) can be arbitrarily chosen in a neighbourhood of the origin, and hence be changed for its opposite.
(iv) Naturally, its continuous limit ($h \to 0$) is equation (2).

We shall refer to these conditions as the *naive discretization rules*. The reason for this terminology is that they simply ensure a formal, easily noticeable, similarity between a differential equation and its discretization scheme. They do *not* imply the preservation of the distinctive features of the continuous equation, such as the Painlevé property, linearisability, analytic expression of the general solution, etc.

In the case of equation (1), we have a third-order differential equation which is of first degree in the third-order derivative. Hence, a faithful discrete version of equation (1) is a relation between $\underline{\underline{u}} = u(x-3\,h/2)$, $\underline{u} = u(x-h/2)$, $\overline{u} = u(x+h/2)$, $\overline{\overline{u}} = u(x+3\,h/2)$, of first degree in $\overline{\overline{u}}$ and $\underline{\underline{u}}$, and invariant under $h \mapsto -h$, $\overline{u} \mapsto \underline{u}$, $\overline{\overline{u}} \mapsto \underline{\underline{u}}$.

There is only one discrete version of the term $u'''$ satisfying these conditions:

$$h^{-3}\,(\overline{\overline{u}} - 3\,\overline{u} + 3\,\underline{u} - \underline{\underline{u}})$$

but there are *a priori* three linearly independent discrete equivalents of $u\,u''$; hence this term will be discretized as

$$\begin{aligned}
&\lambda_1\,h^{-2}\tfrac{1}{2}\left(\overline{u}(\overline{\overline{u}} - 2\overline{u} + \underline{u}) + (\overline{u} - 2\underline{u} + \underline{\underline{u}})\underline{u}\right)\\
&\quad+\lambda_2\,h^{-2}\tfrac{1}{2}\left(\underline{u}(\overline{\overline{u}} - 2\overline{u} + \underline{u}) + (\overline{u} - 2\underline{u} + \underline{\underline{u}})\overline{u}\right)\\
&\quad+\lambda_3\,h^{-2}\tfrac{1}{2}\left(\underline{\underline{u}}(\overline{\overline{u}} - 2\overline{u} + \underline{u}) + (\overline{u} - 2\underline{u} + \underline{\underline{u}})\overline{\overline{u}}\right)
\end{aligned}$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Similarly, the term $u'^2$ possesses three valid discrete equivalents, and will be discretized as

$$\mu_1\,h^{-2}\,(\overline{u} - \underline{u})^2 + \mu_2\,h^{-2}\,(\overline{\overline{u}} - \overline{u})\,(\underline{u} - \underline{\underline{u}}) + \mu_3\,h^{-2}\,(\overline{\overline{u}} - \underline{u})\tfrac{1}{4}(\overline{u} - \underline{\underline{u}})$$

with $\mu_1 + \mu_2 + \mu_3 = 1$. We thus obtain an expression $E$ whose continuous limit is $C$, the left-hand side of equation (1). We notice that $E$ depends solely on the two parameters

$$\mu_1' = \mu_1 - \tfrac{1}{3}(2\,\lambda_1 + 3\,\lambda_2) + \tfrac{2}{9} \qquad \mu_2' = \mu_2 - \tfrac{1}{3}(\lambda_2 - 2\lambda_1)$$

namely the naive discretization of (1) is

$$\begin{aligned}
E \equiv\ & h^{-3}\left(\overline{\overline{u}} - 3\,\overline{u} + 3\,\underline{u} - \underline{\underline{u}}\right)\\
&+\tfrac{1}{12}h^{-2}\left(16\,(\overline{\overline{u}}\,\underline{u} + \overline{u}\,\underline{\underline{u}}) - 3\,\overline{u}\,\underline{u} - 27\,\overline{\overline{u}}\,\underline{\underline{u}} - \overline{\overline{u}}\,\overline{u} - \underline{u}\,\underline{\underline{u}}\right)\\
&+\tfrac{3}{4}\mu_1'\,h^{-2}\left(4\,(\overline{\overline{u}}\,\underline{u} + \overline{u}\,\underline{\underline{u}}) - 3\,\overline{u}\,\underline{u} - 3\,\overline{\overline{u}}\,\underline{\underline{u}} - \overline{\overline{u}}\,\overline{u} - \underline{\underline{u}}\,\underline{u}\right)\\
&+\tfrac{3}{4}\mu_2'\,h^{-2}\left(\overline{\overline{u}}\,\underline{\underline{u}} + 4\,(\overline{u}^2 + \underline{u}^2) - 7\,\overline{u}\,\underline{u} - \overline{\overline{u}}\,\overline{u} - \underline{\underline{u}}\,\underline{u}\right) = 0\,. \qquad (4)
\end{aligned}$$

## 3. Discrete Painlevé test

Consider an $N$th-order finite-difference equation like equation (3), depending on a step $h$. We say this equation has the *(discrete) Painlevé property* iff its general solution

$x \mapsto u(x; h\ldots)$ is free from movable critical points in the $x$ plane, provided $h$ belongs to a suitable neighbourhood of the origin.

Suppose that the finite-difference equation under consideration possesses a continuous ($h \to 0$) limit like equation (2). Then we can apply to it a discrete Painlevé test, called the *method of perturbation of the continuous limit*. It was originally set up by Conte and Musette in [2], and is an analogue of the perturbative Painlevé test for continuous equations [3].

This test consists of making a perturbative expansion of the general solution $u$ of (3) in function of an *a priori* extraneous parameter $\varepsilon$. This induces a similar expansion of $E$:

$$u = \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)} \qquad E = \sum_{n=0}^{+\infty} \varepsilon^n E^{(n)}$$

which has the following property: all equations $E^{(n)} = 0$, $n \geqslant 1$ are the same linear equation with different right-hand sides, namely

$$E^{(n)} \equiv \left\langle \mathrm{d}E^{(0)}, u^{(n)} \right\rangle + R^{(n)}(u^{(0)}, \ldots u^{(n-1)}) = 0$$

where $\langle \mathrm{d}E^{(0)}, u^{(n)} \rangle$ denotes the differential (or Gâteaux derivative) of $E^{(0)}$, taken at $u = u^{(0)}$, acting on the test function $u^{(n)}$.

The choice $\varepsilon = h$ has the extra property that $E^{(0)} = \mathcal{E}$, the continuous limit of $E$. Then, a necessary condition for equation (3) to possess the Painlevé property is that the general solution $u^{(n)}$ of every equation $E^{(n)} = 0$ be free from movable critical singularities.

Practically, we seek all possible Laurent series representations

$$u = \sum_{n=0}^{+\infty} \varepsilon^n \sum_{j=\rho n}^{+\infty} u_j^{(n)} \chi^{j+p} \qquad \chi = x - x_0$$

where $\rho$ is the least Fuchs index of the linearized zeroth-order equation $\mathrm{d}E^{(0)} = 0$. In this expansion a free parameter, $u_j^{(n)}$, enters at each order $n$ of perturbation every time $j$ is a Fuchs index of $\mathrm{d}E^{(0)}$; and the Painlevé property implies that all the corresponding coefficients $E_j^{(n)}$ of $\chi^{j+p}$ in $E^{(n)}$ are zero.

We have applied this test to equation (4). At order 0, we get equation (1):

$$E^{(0)} \equiv u^{(0)'''} - 2\, u^{(0)}\, u^{(0)''} + 3 \left(u^{(0)'}\right)^2 = 0$$

which admits the solution $u = -6\,\chi^{-1}$. Then the Fuchs indices of

$$\mathrm{d}E^{(0)} \equiv \partial^3 - 2\, u^{(0)}\, \partial^2 - 2\, u^{(0)''}\, \mathbf{1} + 6\, u^{(0)'}\, \partial$$

are $-3$, $-2$, $-1$. At perturbation order one, we have $E^{(1)} \equiv \langle \mathrm{d}E^{(0)}, u^{(1)} \rangle = 0$, whose solution is chosen as $u^{(1)} = \left(u_{-3}^{(1)}\, \chi^{-3} + u_{-2}^{(1)}\, \chi^{-2}\right) \chi^{-1}$.

At perturbation order two, we get the condition

$$E_{-2}^{(2)} \equiv \mu_2' = 0\,.$$

Then, if this condition is satisfied, the general solution of equation (4) is free from movable critical singularities up to perturbation order 16 at least, and most probably up to infinity. Thus, there are great chances that equation (4) has the Painlevé property when $\mu_2' = 0$.

This condition $\mu_2' = 0$ is also the only one given by the singularity confinement criterion of Grammaticos *et al* (see [4] for this test).

## 4. Two-parameter solutions

Singularity analysis has given us a one-parameter family of acceptable discrete versions of equation (1), namely equations (4) with $\mu_2' = 0$. But the examination of the more specific properties of equation (1) leads us to restrict our choice, because not all equations (4) preserve these features even if $\mu_2' = 0$.

For instance, equation (1) admits a two-parameter particular solution

$$u(x) = -6 \frac{x - c_1}{(x - c_2)^2} \tag{5}$$

with $c_1$, $c_2$ arbitrary in the complex plane. Demanding the preservation of the solution (5) for any $c_1$ and $c_2$ yields that the two conditions $\mu_1' = 0$ *and* $\mu_2' = 0$ should be satisfied. Hence, the most faithful discretization scheme is

$$\gamma \equiv h^{-3} \left(\overline{\overline{u}} - 3\,\overline{u} + 3\,\underline{u} - \underline{\underline{u}}\right) + \tfrac{1}{12}\, h^{-2} \left(16\,(\overline{\overline{u}}\,\underline{u} + \overline{u}\,\underline{\underline{u}}) - 3\,\overline{u}\,\underline{u} - 27\,\overline{\overline{u}}\,\underline{\underline{u}} - \overline{\overline{u}}\,\overline{u} - \underline{u}\,\underline{\underline{u}}\right) = 0\,. \tag{6}$$

Eliminating $c_1$ and $c_2$ between $u(x)$ calculated by (5) and its first two derivatives yields the least-degree second-order differential equation satisfied by (5):

$$S \equiv 9\,u''^2 + 2\,(u^2 - 9\,u')\,u\,u'' + 3\,(8\,u' - u^2)\,u'^2 = 0\,. \tag{7}$$

The link between equations (1) and (7) is

$$S' - 2\,u\,S = 2\,(u^3 - 9\,u\,u' + 9\,u'')\,C \tag{8}$$

and equation (5) is not a solution of $u^3 - 9\,u\,u' + 9\,u'' = 0$.

Let us examine how we can faithfully discretize equations (7) and (8). By the naive discretization rules, a discrete version of equation (7) should be a relation between $u = u(x)$, $\overline{u} = u(x+h)$ and $\underline{u} = u(x-h)$ of second degree in $\overline{u}$ and $\underline{u}$ and invariant under $h \mapsto -h$, $\overline{u} \mapsto \underline{u}$.

Eliminating $c_1$ and $c_2$ between $u$, $\overline{u}$ and $\underline{u}$ calculated by equation (5), we get the following infinite-order discretization scheme:

$$9\,h^{-4}\,(\overline{u} - 2\,u + \underline{u})^2 + 3\,h^{-3}\,(\overline{u} - \underline{u})\,(2\,u^2 + u\,(\overline{u} + \underline{u}) - 4\,\overline{u}\,\underline{u})$$
$$+h^{-2}\left(\tfrac{1}{4}\,u^2\,(\overline{u}^2 + \underline{u}^2) - 2\,u\,\overline{u}\,\underline{u}\,(\overline{u} + \underline{u}) + 4\,\overline{u}^2\,\underline{u}^2 - \tfrac{1}{2}\,u^2\,\overline{u}\,\underline{u}\right) = 0 \tag{9}$$

whose general solution is automatically (5). We check that equation (9) satisfies the naive discretization rules.

As for the discretization of equation (8), we must pay attention to the fact that equations (6) and (9) do not involve the same values of $u$, despite the misleading use of similar notation. To transpose equation (9) into the world of third-order finite-difference equations, we can think of two possibilities:

- $\overline{\sigma}$ defined as the left-hand side of equation (9) shifted to the right by half a step, i.e. formally $\overline{u} \mapsto \overline{\overline{u}}$, $u \mapsto \overline{u}$, $\underline{u} \mapsto \underline{u}$.
- $\underline{\sigma}$ defined as the left-hand side of equation (9) shifted to the left by half a step, i.e. formally $\overline{u} \mapsto \overline{u}$, $u \mapsto \underline{u}$, $\underline{u} \mapsto \underline{\underline{u}}$.

Then the four-point discretization of the left-hand side $S' - 2\,u\,S$ of equation (8) must obey the naive scheme

$$\omega = (\overline{\sigma} - \underline{\sigma})/h - \lambda_1(\overline{u}\,\overline{\sigma} + \underline{u}\,\underline{\sigma}) - \lambda_2(\underline{u}\,\overline{\sigma} + \overline{u}\,\underline{\sigma}) - \lambda_3(\overline{\overline{u}}\,\overline{\sigma} + \underline{\underline{u}}\,\underline{\sigma}) - \lambda_4(\underline{\underline{u}}\,\overline{\sigma} + \overline{\overline{u}}\,\underline{\sigma})$$

with $\sum_1^4 \lambda_i = 1$. A necessary condition for $\omega$ having the left-hand side $\gamma$ of equation (7) as a factor is that the resultant of $\gamma$ and $\omega$—both being seen as polynomials in $h^{-1}$—be zero. This happens if and only if

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(-\tfrac{4}{3}, \tfrac{1}{12}, 0, \tfrac{9}{4}\right) .$$

In that case, we check that $\omega = f\,\gamma$, with the factor

$$f = \tfrac{1}{4}\Big(\overline{\overline{u}}\,\overline{u}^2 + 16\,(\overline{\overline{u}}\,\underline{u}^2 + \overline{u}\,\underline{\underline{u}}^2) - 8\,\overline{u}\,\underline{u}\,(\overline{\overline{u}} + \underline{\underline{u}}) - 5\,\overline{u}\,\underline{u}\,(\overline{u} + \underline{u}) + \underline{u}^2\,\underline{\underline{u}}\Big)$$
$$+3\,h^{-1}\Big(\overline{\overline{u}}\,\overline{u} - \overline{u}^2 - 4\,(\overline{\overline{u}}\,\underline{u} - \overline{u}\,\underline{\underline{u}}) + \underline{u}^2 - \underline{u}\,\underline{\underline{u}}\Big) + 9\,h^{-2}\Big(\overline{\overline{u}} - \overline{u} - \underline{u} + \underline{\underline{u}}\Big)$$

having as its continuous limit $2\,(u^3 - 9\,u\,u' + 9\,u'')$, i.e. the proportionality factor between $S' - 2\,u\,S$ and $C$ in equation (8).

## 5. Conclusion

The example treated in this letter has shown the efficiency of the method of perturbation of the continuous limit. It has recovered the same condition as that given by the singularity confinement criterion. While the confinement test seems essentially discrete, this perturbative method admits as its continuous limit the continuous Painlevé test.

The integration of the Chazy equation in terms of solutions to the (linear) hypergeometric equation was performed by Chazy [1] and Bureau [5]. For the discrete analogue (6), this remains an open problem. Indeed, the integration process of the continuous equation involves an exchange of the dependent and independent variables, a feature which seems hard to transpose into the discrete world.

Then the perservation of the two-parameter solution (5) appears as a minimal demand. It leaves only one possibility, which thus may be called the 'most faithful' finite-difference equation representing the Chazy equation.

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